MATH 54 – MIDTERM 2 – BONUS

PEYAM RYAN TABRIZIAN

Name: ____

Instructions: This bonus question (the same from Midterm 2) is *optional* and is due on Monday, July 16th, at 2 pm in **4 Evans**! (our usual classroom). It counts for 1 point. You are **not** allowed to collaborate with others and **not** allowed to use the textbook or other outside resources (like google) to solve it!

Bonus In this problem, we're going to use the Wronskian to find the general solution of a quite complicated differential equation! This should illustrate yet again the power of the Wronskian!

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian (determinant):

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

Calculate $W'(t)$ and express your answer in terms of $W(t)$.

(see hints on the next page)

Date: Monday, July 16th, 2012.

Hints:

1) You **HAVE** to use the fact that $y_1'' = -P(t)y_1' - Q(t)y_1$ and $y_2'' = -P(t)y_2' - Q(t)y_2$.

2) Also recall that $y_2'y_1 - y_1'y_2 = W(t)$

(b) Solve the differential equation found in (a).

Hint: The general solution to $y' = f(t)y$ is $y(t) = Ce^{\int f(t)dt}$

Note: From now on, ignore the constants, i.e. in your answer in (b), set $C = 1$

(c) From (b), we get:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \underline{\hspace{4cm}} \text{(your answer from } (b)\text{)}$$

Solve for y_2 in terms of y_1.

Hints:

1) Divide this equality by $(y_1(t))^2$
2) Recognize the left-hand-side as the derivative of $\frac{y_2}{y_1}$
3) Integrate, and solve for y_2 in terms of y_1. You answer will involve another $\int$ sign! Again, ignore the constants!

(d) Let's apply the result in (c) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $\boxed{y_1(t) = \sin(t)}$.

Use your answer in (b) to find *another* solution $y_2(t)$!

Note: Again, you can ignore the constants!

Hint: You should use the following facts, in order:

1) $\int \tan(t)dt = -\ln(\cos(t))$
2) $e^{-a} = \frac{1}{e^a}$ and $e^{\ln(a)} = a$
3) At some point, multiply your integrand (the fct you're integrating) by $\frac{\cos(t)}{\cos(t)}$
4) $\cos^2(t) = 1 - \sin^2(t)$
5) The substitution $u = \frac{1}{\sin(t)}$, then $du = \frac{-\cos(t)}{\sin^2(t)}$ and $\sin(t) = \frac{1}{u}$
6) $\frac{1}{1-\frac{1}{u^2}} = \frac{u^2}{u^2-1}$ (multiply top and bottom by u^2)
7) The formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$
8) The formula $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$
9) The formula $\frac{1}{2}\ln\left|\frac{1-z}{1+z}\right| = \coth^{-1}(z)$
10) Try to have a formula *without* $\frac{1}{\sin(t)}$, and for this use the fract that $\frac{\frac{1}{z}-1}{\frac{1}{z}+1} = \frac{1-z}{1+z}$

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the *general* solution of $y'' - \tan(t)y' + 2y = 0$? (no need to show your work here)